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Modulational Instability of Wave Trains in a Rotating Ocean

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Abstract

The problem of modulational stability of quasi-monochromatic wave-trains propagating in a rotating fluid which provides both the small-scale Boussinesq dispersion and large-scale Coriolis dispersion is studied. We derive two-dimensional non-linear Schrödinger (NLS) equation from the basic set of Boussinesq equations for shallow water waves taking into account the Coriolis force caused by Earth' rotation. For unidirectional waves propagating in one direction only the considered set of equations reduces to the Gardner–Ostrovsky equation which is applicable only within a finite range of wavenumbers. It is shown that the narrow-band wave-trains are modulationally stable for relatively small wavenumbers $k < k_c$ and unstable for $k > k_c$, where k_c is some critical wavenumber. The derived NLS equation is applicable even for waves with very small wavenumbers up to zero. The detailed analysis of coefficients of the NLS equation is presented both for one-dimensional and two-dimensional cases. The conditions of self-modulation and self-focussing are determined and presented graphically on the plane of parameters. Application of results obtained to the real oceanic conditions is discussed.

Introduction

As well-known, small-amplitude quasi-monochromatic surface and internal gravity waves on a shallow water are stable with respect to self-modulation and self-focussing [1]. This result formally agrees with what follows from the Korteweg–de Vries (KdV) equation:

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \alpha\eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0, \qquad (1)$$

where the nonlinear coefficient α for internal waves can be of either sign, and $\beta > 0$ both for surface and internal gravity waves. In particular, for surface gravity waves the coefficients are [1, 2] $\alpha = 3c_0/2h$, $\beta = c_0h^2/6$, where $c_0 = (gh)^{1/2}$ is the speed of long linear waves, g is the acceleration due to gravity, h is the fluid depth.

The dispersion relation between the wave frequency ω and wavenumber k for infinitesimal-amplitude waves in the linearised Eq. (1) is well-known (see, e.g., [1, 6]): $\omega(k) = c_0k - \beta k^3$. It shows that the dispersion appears at relatively large k (small wavelength $\lambda = 2\pi/k$), when the influence of the second term in the right-hand side is not negligibly small. Such small-scale ("Boussinesq-type") dispersion is typical for long water waves. The derivation of Eq. (1) is based on the assumption that the dispersion is weak, so that $\beta k^3 << c_0k$ or $k << k_B$, where $k_B = (c_0/\beta)^{1/2}$.

When the Earth' rotation is taken into account the large-scale ("Coriolis-type") dispersion appears, which manifests when $k \to 0$ and asymptotically disappears when $k \to \infty$ [5, 10, 11]. In the intermediate range of wavelength (wave number) the large-scale dispersion is small and of the same order of smallness as the small-scale dispersion, so that the combined dispersion relation can be presented in the form $\omega(k) = c_0k - \beta k^3 + \gamma/k$, where

$$\sqrt{\gamma/c_0} << k << \sqrt{c_0/\beta}, \qquad (2)$$

$\gamma > 0$ is a constant. The corresponding weakly nonlinear evolution equation generalising the KdV equation (1) is known as the Ostrovsky equation [5, 10, 11]:

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \alpha\eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} \right) = \gamma\eta. \qquad (3)$$

On the basis of this equation it was shown [3, 4, 14, 15] that the large scale dispersion drastically changes the modulation stability of quasi-harmonic wave-trains. The nonlinear correction to the wave frequency remains negative for all wavenumbers as in the case of KdV equation, but the dispersion coefficient in the non-linear Schrödinger equation changes its sign at $k = k_c \equiv (\gamma/3\beta)^{1/4}$ when the group velocity $V_g = d\omega/dk$ attains maximum. Therefore, the corresponding NLS equation according to the Lighthill criterion [1, 6, 8, 12, 17] remains modulationally stable for $k < k_c$, but becomes unstable for $k > k_c$. This is in a sharp contrast with the intuition based on the NLS equation derived from the KdV equation (1).

As the Ostrovsky equation is an approximate one and has a limited range of validity, the issue of modulation stability of wave-trains remained uncertain thus far, because at very small wavenumbers where this model equation is inapplicable the situation with the modulation stability could be different. In addition to that the Ostrovsky equation is one-dimensional and cannot describe two-dimensional wave process depending on two special variables. Therefore the problem of modulation stability of wave-trains should be reconsidered within the framework of more accurate equations in the long-wave limit. Moreover, for the analysis of modulation stability of water waves both the KdV and Ostrovsky model equations are, obviously, insufficient, because they contain only the quadratic nonlinear terms, whereas the cubic nonlinear terms in the primitive equations usually provide the same order contribution to the nonlinear coefficient of NLS equation.

Below the problem of modulation stability of quasi-monochromatic wave-trains is studied on the basis of two-dimensional shallow-water model set of equations derived by Shrira [5, 13]. We derive the 2D NLS equation from this set of equation and study a stability of quasi-monochromatic wave-trains with respect to longitudinal and transverse modulations. Then we compare the results obtained with what follows from the NLS equation derived from the model Ostrovsky equation and its generalisation [3, 4, 14, 15]. We present a diagram on the plane of parameters (components of a wave vector) which illustrate illustrates zones were self-modulation and self-focussing instability can occur.

The governing equations and dispersion relations

We start with the following set of equations applicable (after appropriate scaling) both to surface and internal waves in the Boussinesq approximation [5, 13]:

$$\frac{\partial \eta}{\partial t} + \nabla_\perp \left[(h+\eta)\mathbf{q} \right] = 0, \tag{4}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla_\perp)\mathbf{q} + [\mathbf{f}\times\mathbf{q}] + \frac{c_0^2}{h}\nabla_\perp \eta + \frac{h}{3}\frac{\partial^2 \nabla_\perp \eta}{\partial t^2} = 0, \tag{5}$$

where η is the perturbation of a free surface in a non-stratified fluid or perturbation of an isopycnal surface (surface of equal density) in a stratified fluid, $\mathbf{q} = (u, \upsilon)$ is the depth averaged fluid velocity with two horizontal components, longitudinal u and transverse υ, $\mathbf{f} = f\mathbf{n}$, where $f = 2\Omega\sin\varphi$ is the Coriolis parameter, Ω is the angular frequency of Earth rotation, φ is the local geographic latitude, $\mathbf{n}$ is the unit vector normal to the Earth surface, and $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$.

By introducing new variables $\tau = ft$, $\xi = x\sqrt{3}/h$, $\chi = y\sqrt{3}/h$, one can present Eqs. (4) and (5) in the dimensionless scalar form:

$$\frac{\partial \tilde\zeta}{\partial \tau} + \frac{\partial \tilde u}{\partial \xi} + \frac{\partial \tilde v}{\partial \chi} = -\frac{\partial(\tilde\zeta \tilde u)}{\partial \xi} - \frac{\partial(\tilde\zeta \tilde v)}{\partial \chi}, \tag{6}$$

$$\frac{\partial \tilde u}{\partial \tau} - \tilde v + C^2\frac{\partial \tilde\zeta}{\partial \xi} + \frac{\partial^3 \tilde\zeta}{\partial \tau^2 \partial \xi} = -\tilde u\frac{\partial \tilde u}{\partial \xi} - \tilde v\frac{\partial \tilde u}{\partial \chi}, \tag{7}$$

$$\frac{\partial \tilde v}{\partial \tau} + \tilde u + C^2\frac{\partial \tilde\zeta}{\partial \chi} + \frac{\partial^3 \tilde\zeta}{\partial \tau^2 \partial \chi} = -\tilde u\frac{\partial \tilde v}{\partial \xi} - \tilde v\frac{\partial \tilde v}{\partial \chi}. \tag{8}$$

where $C^2 = 3(c_0/hf)^2$ stands for the normalised characteristic wave speed, and $\tilde u = u\sqrt{3}/hf$, $\tilde v = v\sqrt{3}/hf$, $\tilde\zeta = \eta/h$. For the estimates we put $C = 10^3$ assuming that $h = 3000$ m, $g = 10$ m/s ($c_0 = 173$ m/s), $f = 10^{-4}$ s^{-1}.

For small-amplitude perturbations $\sim\exp[i(\omega t - k_x\xi - k_y\chi)]$ in the linear approximation the following dispersion relation can be derived from Eqs. (6)–(8):

$$\omega^2 = \frac{C^2k^2+1}{1+k^2}, \tag{9}$$

where $k^2 = k_x^2 + k_y^2$.

Assuming that in general, for waves of small, but finite amplitude A, the wave frequency depends on the wave-number components k_x and k_y and also on A^2, we can present the dispersion relation (9) in terms of the Taylor series around a point $(k_{x0}, k_{y0}, 0)$:

$$\omega \approx \omega_0 + V_{gx}(k_x - k_{x0}) + V_{gy}(k_y - k_{y0}) + \frac{1}{2}\frac{\partial^2\omega}{\partial k_x^2}(k_x - k_{x0})^2 +$$

$$\frac{1}{2}\frac{\partial^2\omega}{\partial k_y^2}(k_y - k_{y0})^2 + \frac{\partial^2\omega}{\partial k_x \partial k_y}(k_x - k_{x0})(k_y - k_{y0}) + \frac{\partial\omega}{\partial M}|A|^2, \tag{10}$$

where $V_{gx} = \partial\omega/\partial k_x$, $V_{gy} = \partial\omega/\partial k_y$ are the components of group velocity $\mathbf{V}_g = (V_{gx}, V_{gy})$ calculated in the point $(k_{x0}, k_{y0}, 0)$; all other partial derivatives are also calculated in the same point, and $M \equiv |A|^2$.

Using the well-known approach [2], one can restore the partial differential equation which corresponds to the approximate dispersion relation (10):

$$i\left(\frac{\partial A}{\partial \tau} + \mathbf{V}_g\cdot\nabla_\perp A\right) +$$

$$p_x\frac{\partial^2 A}{\partial \xi^2} + p_y\frac{\partial^2 A}{\partial \chi^2} + 2p_{xy}\frac{\partial^2 A}{\partial \xi \partial \chi} + q|A|^2 A = 0, \tag{11}$$

where

$$p_x = \frac{1}{2}\frac{\partial^2\omega}{\partial k_x^2}, \quad p_y = \frac{1}{2}\frac{\partial^2\omega}{\partial k_y^2}, \quad p_{xy} = \frac{1}{2}\frac{\partial^2\omega}{\partial k_x \partial k_y}, \quad q = -\frac{\partial\omega}{\partial M}.$$

Making the orthogonal transformation of coordinates with the new X-axis along the vector of group velocity and Y-axis in the perpendicular direction, one can present this equation in the standard form of the two-dimensional non-linear Schrödinger equation:

$$i\frac{\partial A}{\partial \tau} + p_x\frac{\partial^2 A}{\partial X^2} + p_y\frac{\partial^2 A}{\partial Y^2} + q|A|^2 A = 0, \tag{12}$$

where $X = \xi - V_{gx}\tau$, $Y = \chi - V_{gy}\tau$, and the coefficients p_x and p_y can be readily found from the dispersion relation (9):

$$p_x = \frac{C^2-1}{2}\frac{1-2k^2-3C^2k^4}{(1+k^2)^{5/2}(1+C^2k^2)^{3/2}}, \tag{13a}$$

$$p_y = \frac{C^2-1}{2}\frac{1}{(1+k^2)^{3/2}(1+C^2k^2)^{1/2}}, \tag{13b}$$

where $k \equiv k_x$ in this coordinate frame and $k_y = 0$.

The dependences of normalised dispersion coefficients $2p_x/(C^2 - 1)$ and $2p_y/(C^2 - 1)$ on k are shown in figure 1 for $C = 10^3$.

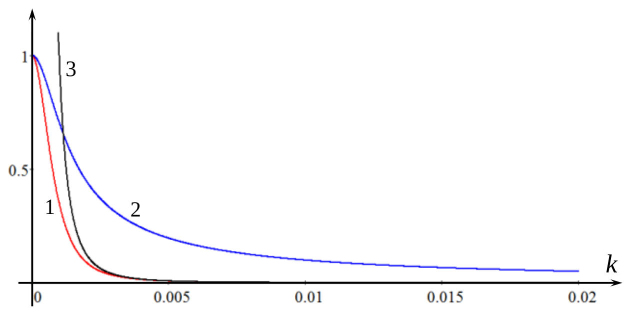

Figure 1. The dependences of normalised dispersion coefficients in the NLS equation (12) as per Eqs. (13). Lines 1 and 2 pertain to p_x and p_y when the NLS equation is derived from Eqs. (6)–(8); line 3 pertains to the normalised dispersion coefficients p_O when the NLS equation is derived from the one-dimensional Ostrovsky Eq. (3).

As one can see, the dispersion coefficient p_y is positive and monotonically decreases with k, whereas the coefficient p_x decreases non-monotonically and changes its sign at the wave number

$$k_1 = \sqrt{\frac{\sqrt{(3C^2+1)}-1}{3C^2}} \approx 0.024\cdot$$

The dispersion coefficient p_O changes its sign at $k = k_O \equiv 1\big/\sqrt[4]{3C^2}$. For very large value of the parameter C ($C >> 1$), $k_1 \approx k_O$.

The nonlinear coefficient q in Eq. (12) has been derived in [9]:

$$q(k) = \frac{3k^2\sqrt{1+C^2k^2}\,(5-C^2+4C^2k^2)}{(1+k^2)^{3/2}(1+5k^2+4C^2k^4)}. \tag{14a}$$

Graphic of this dependence is shown in figure 2 by line 1 for $C = 10^3$. The nonlinear coefficient (14) is non-monotonic and changes its sign at $k_2 \equiv \sqrt{C^2-5}\big/2C \approx 0.5\cdot$

If the NLS equation is derived from the one-dimensional Ostrovsky Eq. (3), then the nonlinear coefficient reads:

$$q_O(k) = \frac{-3C^3k^3}{1+4C^2k^4}. \qquad (14b)$$

The dependence $q_O(k)$ is non-monotonic too, but $q_O(k)$ does not change its sigs and remains negative for all k (line 2 in figure 2).

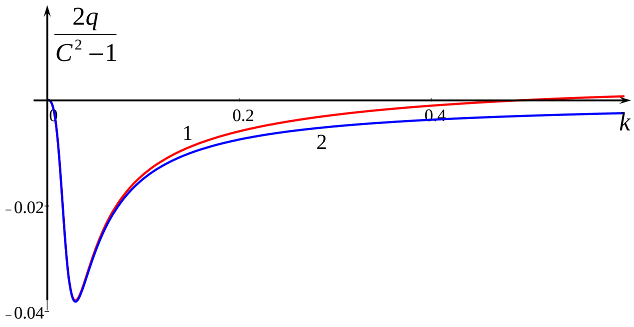

Figure 2. The dependences of normalised nonlinear coefficients in the NLS equation (12). Line 1 pertains to the case when the NLS equation is derived from Eqs. (6)–(8) as per Eq. (14a); line 2 pertains to the case when the NLS equation is derived from the Ostrovsky Eq. (3) as per Eq. (14b).

The signs of the dispersive and nonlinear coefficients are very important because they determine the stability or instability of wave trains with respect to self-modulation or self-focussing.

The analysis of modulational instability

As well known, the stability of quasi-monochromatic wave trains with respect to small modulations is determined within the framework of NLS equation (12) by the relative sign of nonlinear and dispersive coefficients. According to the Lighthill criterion [1, 6, 8, 12, 17], the stability with respect to self-modulation occurs when $p_x(k)q(k) < 0$, otherwise, when $p_x(k)q(k) > 0$, the wave-trains are unstable. In the latter case the "bright" envelope solitons can exist as well as the breathers – nonstationary solitary waves which oscillate ("breathe") in the process of propagation [11]. In the former case only "dark" solitons can exist on the background of a sinusoidal wave (see the references cited above).

In one-dimensional case the analysis of wave-train stability with respect to self-modulation was performed in [9]; the result is shown in figure 3.

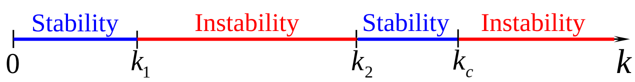

Figure 3. Ranges of stability and instability of wave trains against self-modulation [9].

As one can see from this figure, the stability of wave trains with respect to self-modulation occurs at small and large wave numbers, i.e. when $0 < k < k_1$, and when $k > k_2$, (notice that the NLS model based on Eqs. (3), (4) is applicable only for relatively small dimensionless wave numbers $k << 1$). The instability occurs in the range $k_1 < k < k_2$. As well-known, in a non-rotating fluid of a finite depth wave trains are stable against self-modulation for $k < k_c \equiv 1.363$ [1], and for $k > k_c$ the self-modulation again occurs (see figure 3).

The model based on the one-dimensional Ostrovsky Eq. (3) predicts the existence of only one boundary between the stability and instability with respect to self-modulation at $k = k_O \approx k_1$.

In the case when the NLS equation is derived from the one-dimensional KdV equation (1) the product $p(k)q(k) = -27C^2/2$ is always negative, therefore wave-trains are stable against self-modulations for all k within the range of validity of the KdV equation $k << 1$.

The wave-train stability with respect to the self-focussing is determined similarly by the product $p_y(k)q(k)$. From Eqs. (13b) and (14a) it follows that the self-focussing can occur only when $k > k_2$, when the nonlinear coefficient becomes positive together with $p_y(k)$ (see Figs. 1 and 2).

In the general case the analysis based on the Lighthill criterion shows that the modulation instability of long waves in a rotating fluid occurs for oblique perturbations propagating under the angle θ with respect to X-axis within a certain sector. The boundaries of the instability region are determined by equation:

$$F(k,\theta) \equiv \left(C^2 - 5 - 4C^2k^2\right)\left[\left(1-2k^2-3C^2k^4\right)+\left(1+k^2\right)\left(1+C^2k^2\right)\tan^2\theta\right] = 0.$$

Figure 4 illustrates the zones where the modulation instability can occur (zones 3 and 4) and where it cannot occur (zones 1 and 2). This figure generalises the diagram shown in figure 3 for one-dimensional case and reduces to figure 3 when $\theta = 0$.

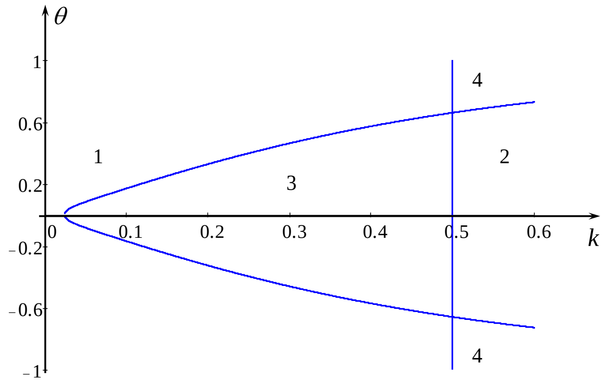

Figure 4. Parameter plane which shows that the modulation instability can occur within the zones 3 and 4, whereas it cannot occur within the zones 1 and 2.

Thus on basis of this analysis we discovered the conditions when the self-modulation and self-focussing phenomena can occur in a rotating shallow water. There is no situations when these two phenomena co-exist, therefore a wave collapse [16] cannot occur in a shallow rotating fluid.

Conclusions

In this paper we have presented a derivation of two-dimensional NLS equation from the basic set of equations describing long waves in a rotating shallow water [5, 13]. In one-dimensional case when all variables depend of only one special coordinate the derived NLS equation agrees well with the NLS equation derived from the Ostrovsky equation (3) [3, 4], as well as from its generalised version, the Gardner–Ostrovsky equation with the cubic nonlinearity [14, 15], within the range of validity of both these model equations. Beyond the range of validity of these equations at relatively small wavelengths ($k > k_2$) and relatively large wavelengths ($0 < k < k_1$) the NLS equation derived from the basic set of shallow-water equations predicts the stability of wave trains with respect to self-modulation, whereas such conclusion cannot be obtained from the model Ostrovsky equation or Gardner–Ostrovsky equation.

The developed theory predicts wave train stability with respect to self-focussing for relatively large wavelength with $k < k_2$.

Notice in the conclusion that similar analysis can be performed not only with respect to water waves, but also to plasma waves, waves in solids, optical fibres, and so on (see [9]).

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